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Missing terms in Classical Inequalities

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1 Historical remarks

先ず、古典的な2つの有名な不等式を思い出そう。一番目はハーディの不等式と呼ばれている。

Hardy inequality;

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^2} dx, \qquad u \in C_0^\infty(\mathbb{R}^n)$$

次の不等式は最も単純なソボレフ型の不等式である。

Sobolev inequality;

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq S \left(\int_{\mathbb{R}^n} |u(x)|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \qquad u \in C_0^\infty(\mathbb{R}^n)$$

次元は $n \geq 3$ とする。ここで u は $u \in W_0^{1,2}(\mathbb{R}^n)$ で十分である。

これらは最もよく知られた古典的不等式の例である。これらの不等式は長い間シャープであると考えられてきたが、実はさらに改良できることが様々な研究で明らかになってきた。ここでは、歴史的に重要と思われる結果を簡単にみていこう。最初の例は、両方が改良できる可能性を示唆している点で面白いものである。

1. V.G. Mazja (85) (in his book Sobolev spases)

$$z = (x, y) \in \mathbb{R}^m \times \mathbb{R}^n, \quad u \in C_0^\infty(\mathbb{R}^{m+n})$$

$$\int_{\mathbb{R}^{m+n}} |\nabla_z u(x,y)|^2 \, dz \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^{m+n}} \frac{u(x,y)^2}{|y|^2} dz$$

$$+C \left(\int_{\mathbb{R}^{m+n}} |u(x,y)|^{\frac{2(m+n)}{m+n-2}} \, dz \right)^{\frac{m+n-2}{m+n}} \qquad ({}^\exists C > 0).$$

これはある種の重み付きソボレフ不等式の系として得られるもので、ソボレフ不等式とハーディ不等式の混合型という意味で興味深いものであった。

次は少し遡るが、本当の意味でミッシングタームの重要性が理解できる例である。

2. H. R.Brezis, L. Nirenberg (83)

次の境界値問題の可解性に関する深い考察が行われた。

$$\begin{cases} -\Delta u - \lambda u = \mu |u|^{\frac{n+2}{n-2}} u, \text{ in } \Omega \quad \text{(bounded domain)}, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

この問題は次の変分問題に帰着することになる。

$$S_\lambda =$$
$$\inf \left[\int_\Omega |\nabla u|^2 \, dx - \lambda \int_\Omega u^2 \, dx : \int_\Omega |u|^{\frac{2n}{n-2}} \, dx = 1, u \in C_0^\infty(\Omega) \right]$$
$$S = S_0; \text{ Sobolev best constant}$$

ここで最も重要な問題は次の問である。

いつ $S_\lambda < S$ が成立するのか?

答は $n \geq 4, \quad \lambda > 0; \quad n = 3, \quad \lambda \geq {}^\exists \lambda_0 > 0$ という驚くべきものであった。

つまり、$\Omega \subset \mathbb{R}^3$; a bounded open set として

$$\int_\Omega |\nabla u|^2 \, dx \geq S \left(\int_\Omega u^6 \, dx \right)^{\frac{1}{3}} + \lambda^* \int_\Omega u^2 \, dx$$

がある正数 λ^* で成り立つのである。この不等式はシャープであり、

$$S_{\lambda^*} = S$$

が成り立っている。さらに

$N \geq 3$, $\Omega \subset \mathbb{R}^3$; a bounded open set, $1 \leq q < \frac{n}{n-2}$ として

$$\int_\Omega |\nabla u|^2 \, dx \geq S \left(\int_\Omega u^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}} + \lambda_q \left(\int_\Omega u^q \, dx \right)^{\frac{2}{q}}$$

が成り立つことも知られている。ここで

$$\frac{n}{n-2} < 2, \quad \text{if} \quad n \geq 3$$

に注意しよう。

2. Horiuchi (96)

$$\int_{\mathbb{R}^n} |\nabla u|^2 |x|^{2\alpha} \, dx \geq$$
$$S\biggl(\int_{\mathbb{R}^n} |u|^q |x|^{q\alpha} \, dx \biggr)^{2/q} + \alpha(\alpha + n - 2) \int_{\mathbb{R}^n} u^2 |x|^{2(\alpha-1)} \, dx$$

ここで S is the Sobolev best constant, $q = \frac{2n}{n-2}$, $\alpha > 1 - \frac{n}{2}$

最良定数をもつ全空間におけるソボレフ不等式はミッシングタームを持たないと長く信じられてきたが、この例はその問題の反例を与えているのであった。

この不等式は次のソボレフ不等式を一般化した重み付きソボレフ不等式に関連する変分問題と密接に関連していることは言うまでもない。

$$S(p, q, \alpha, \beta, n) =$$
$$= \inf \left[\int_{\mathbb{R}^n} |\nabla u|^p |x|^{p\alpha} \, dx : u \in C_0^\infty(\mathbb{R}^n), \int_{\mathbb{R}^n} |u|^q |x|^{\beta q} \, dx = 1 \right].$$

where

$$\left\{ \begin{array}{l} 1 < p < +\infty, \quad (1 - \alpha + \beta)p < n, \quad n \geq 2, \\ 0 < 1/p - 1/q = (1 - \alpha + \beta)/n \\ -n/q < \beta \leq \alpha. \end{array} \right.$$

関連する結果を一つあげておこう。

Theorem 1.1 *(T. Horiuchi) (1) Assume that $0 < \alpha = \beta$, $1/2 - 1/q = 1/n$, $n > 2$. Then it holds that*

$$S(2, q, \alpha, \alpha, n) = S(2, q, 0, 0, n) = S.$$

Moreover there exists no extremal function which attains the infimum in $W^{1,2}_{\alpha,\alpha}(\mathbb{R}^n)$.

(2) *Assume that $\alpha > 0, \alpha > \beta, 0 < 1/p - 1/q = (1 - \alpha + \beta)/n, n \geq 2$ and $1 < p < \frac{n}{1-\alpha+\beta}$.*
Then the infimum $S(p, q, \alpha, \beta, n)$ is attained by an extremal function u in $W^{1,p}_{\alpha,\beta}(\mathbb{R}^n)$ and this u satisfies in distribution sense the equation:

$$-\operatorname{div}(|x|^{p\alpha} |\nabla u|^{p-2} \nabla u) = I \cdot |x|^{\beta q} |u|^{q-2} u,$$

where I is a Lagrange multiplier.

4. H. R. Brezis and J. L. Vazquez (97)

$1 \le q < \frac{2n}{n-2}$, $C_q(|\Omega|) > 0$, $N \ge 3$ such that

$$\int_\Omega |\nabla u|^2 dx \ge \frac{(n-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx + C_q(|\Omega|) \left(\int_\Omega |u|^q dx \right)^{\frac{2}{q}},$$

$\forall u \in W_0^{1,2}(\Omega)$.

彼らは次の半線形楕円型方程式の解の爆発問題を考察した。

$$-\Delta u = \lambda f(u), \qquad \text{in } \Omega,$$

$$u = 0 \qquad \text{on } \partial\Omega$$

ここで f is 凸な正値間数である。

5. T. Horiuchi (02)

$n > 4$. Let Ω be a bounded domain of $\mathbb{R}^n$.
For any $u \in H_0^2(\Omega)$,

$$\begin{aligned} \int_\Omega |\Delta u|^2 \, dx &\ge \left(\frac{n(n-4)}{4}\right)^2 \int_\Omega \frac{|u|^2}{|x|^4} \, dx \\ &+ \lambda_1 \cdot \left(\frac{\omega_n}{|\Omega|}\right)^{\frac{2}{n}} \frac{n(n-4)}{2} \int_\Omega \frac{|u|^2}{|x|^2} \, dx + \lambda_2 \cdot \left(\frac{\omega_n}{|\Omega|}\right)^{\frac{4}{n}} \int_\Omega |u|^2 \, dx \end{aligned}$$

$n > 8$.

For any $u \in H_0^4(\Omega)$

$$\begin{aligned} \int_\Omega |\Delta^2 u|^2 \, dx &\ge H(n, \Delta^2) \int_\Omega \frac{|u|^2}{|x|^8} \, dx \\ &+ a_1 \cdot \lambda_1 \cdot \left(\frac{\omega_n}{|\Omega|}\right)^{\frac{2}{n}} \int_\Omega \frac{|u|^2}{|x|^6} \, dx + a_2 \cdot \lambda_2 \cdot \left(\frac{\omega_n}{|\Omega|}\right)^{\frac{4}{n}} \int_\Omega \frac{|u|^2}{|x|^4} \, dx \\ &+ a_3 \cdot \lambda_3 \cdot \left(\frac{\omega_n}{|\Omega|}\right)^{\frac{6}{n}} \int_\Omega \frac{|u|^2}{|x|^2} \, dx + \lambda_4 \cdot \left(\frac{\omega_n}{|\Omega|}\right)^{\frac{8}{n}} \int_\Omega |u|^2 \, dx. \end{aligned}$$

ここで

$$H(n, \Delta^2) = \left(\frac{n(n-4)(n+4)(n-8)}{16}\right)^2$$

a_1, a_2 と a_3 は次で定められる正定数である。

$$\begin{cases} a_1 = \frac{1}{16} n^2 (n-4)^2 (n+4)(n-8), \\ a_2 = \frac{3}{8} n(n-4)(n+4)(n-8), \\ a_3 = (n+4)(n-8). \end{cases}$$

6. (Adimurthi, Chaudhuri, and Ramaswamy (02))

par Let $R > \sup_\Omega |x|e^{\frac{2}{p}}, \quad n \geq 2, \quad 1 < p < n.$
For any $u \in W_0^{1,p}(\Omega), \exists \quad C > 0$ depending on n, p and R such that

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x|^p} dx + C \int_\Omega \frac{|u(x)|^p}{|x|^p} \left(\log \frac{R}{|x|}\right)^{-\gamma} dx$$

if $\gamma \geq 2$.

When $p = n$ it holds that

$$\int_\Omega |\nabla u|^n dx \geq C \int_\Omega \frac{|u(x)|^n}{|x|^n} \left(\log \frac{R}{|x|}\right)^{-n} dx$$

More references

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2 Coffee break

—————— A sharp Fatou's inequality——————

少しちがった意味のミッシングタームの例を紹介しましょう。

$$f_k, \qquad f \in L^q(\Omega) \qquad (q \geq 1)$$

$$f_k \to f \qquad \text{weakly and a.e.}$$

そのとき次の A sharp Fatou's inequality　と呼ばれるものが成立します。

$$\lim_{k\to\infty} (||f_k||_q^q - ||f_k - f||_q^q) = ||f||_q^q \qquad (E.H.Lieb)$$

これから直ちに
A Fatou's inequality:

$$\lim_{k\to\infty} ||f_k||_q^q \geq || \lim_{k\to\infty} f_k||_q^q$$

を導くことができるわけです。

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3 Introduction

さて本題に戻って、今回の講演で取り上げる不等式を紹介しましょう。

Introduction 1 (Rellich type)

$$\int_\Omega |\Delta u|^p dx \geq \Lambda_{n,p} \int_\Omega \frac{|u(x)|^p}{|x|^{2p}} dx \tag{1}$$

for any $u \in W_0^{2,p}(\Omega)$, where Ω a bounded domain in $\mathbb{R}^n$ with $0 \in \Omega$, $n \geq 3$, and $1 < p < \frac{n}{2}$. Here the best constant

$$\Lambda_{n,p} = n^p \left(\frac{n-2p}{p}\right)^p \left(\frac{p-1}{p}\right)^p$$

is given by the infimum of

$$I(u) = \frac{\int_\Omega |\Delta u|^p dx}{\int_\Omega \frac{|u(x)|^p}{|x|^{2p}} dx}$$

over $W_0^{2,p}(\Omega)$.

Moreover there exists no extremal function in $W_0^{2,p}(\Omega)$ which attainsthe infimum of these problem. In view of this, we shall investigate the Hardy inequalities (1) and improve them by finding out missing terms.

Our aim in this talk is to achieve an optimal improvement of the inequality (1) by adding a second term involving the singular weight $\left(\frac{1}{\log \frac{1}{|x|}}\right)^2$, in the sense that the improved inequality holds for this weight but fails for any weight more singular than this one.

Application

As an application, we use our improved inequality to determine exactly when the first eigenvalue of the weighted eigenvalue problem for the operator

$$L_\mu u = \Delta\left(|\Delta u|^{p-2}\Delta u\right) - \frac{\mu}{|x|^{2p}}|u|^{p-2}u$$

will tend to 0 as μ increases to $\Lambda_{n,p}$.

Introduction 2 (Hardy type)

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x|^p} dx \tag{2}$$

for any $u \in W_0^{1,p}(\Omega)$, where Ω a bounded domain in $\mathbb{R}^n$ with $0 \in \Omega$, $n \geq 3$, and $1 < p < n$.

Here the best constant is given by the infimum of

$$I(u) = \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \frac{|u(x)|^p}{|x|^p} dx}$$

over $W_0^{1,p}(\Omega)$.

When $p = 2$, we shall investigate the Hardy inequalities and improve them by finding out finitely many missing terms.

4 Main results

Main result 1

The notations: For $t > 0$ and $k \geq 2$,

$$A_1(t) := \log \frac{R}{t}, \quad A_k(t) := \log A_{k-1}(t), \quad e_1 := e, \, e_k := e^{e_{k-1}}.$$

Theorem 1 Let $n \geq 2$, $k \geq 1$ and $R \geq e_k \sup_\Omega |x|$. For any $u \in W_0^{1,2}(\Omega)$, there exist sharp remainder terms such that

$$\begin{aligned}\int_\Omega |\nabla u(x)|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u(x)^2}{|x|^2} dx + \\ \frac{1}{4}\int_\Omega \frac{u(x)^2}{|x|^2} \Big[A_1(|x|)^{-2} + \\ \Big(A_1(|x|)A_2(|x|)\Big)^{-2} + \cdots + \\ + \cdots\cdots\cdots + \\ + \Big(A_1(|x|)A_2(|x|)\ldots A_k(|x|)\Big)^{-2}\Big] dx.\end{aligned} \tag{3}$$

Main result 2

Theorem 2 Let $n \geq 3$, $0 \in \Omega$ and Ω is a bounded domain in $\mathbb{R}^n$.

1. Noncritical case ($1 < p < \frac{n}{2}$)
   Assume $\gamma \geq 2$, then there exists $K = K(n) > 0$ and $C = C(n) > 0$ such that if $R > K \sup_\Omega |x|$ then
   $$\begin{aligned}\int_\Omega |\Delta u|^p dx \geq \left(\frac{n-2p}{p}\right)^p \left(\frac{np-n}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x|^{2p}} dx \\ + C\int_\Omega \frac{|u(x)|^p}{|x|^{2p}} \left(\log \frac{R}{|x|}\right)^{-\gamma} dx\end{aligned} \tag{4}$$
   for any $u \in W_0^{2,p}(\Omega)$.
2. Critical case ($p = \frac{n}{2}$)

   Assume $\gamma \geq \frac{n}{2}$, then there exists $K^* = K^*(n) > 0$ and $C^* = C^*(n) > 0$ such that if $R > K^* \sup_\Omega |x|$ then
   $$\begin{aligned}\int_\Omega |\Delta u|^{\frac{n}{2}} dx \geq \left(\frac{n-2}{\sqrt{n}}\right)^n \int_\Omega \frac{|u(x)|^{\frac{n}{2}}}{|x|^n} \left(\log \frac{R}{|x|}\right)^{-\gamma} dx \\ + C^* \int_\Omega \frac{|u(x)|^{\frac{n}{2}}}{|x|^n} \left(\log \frac{R}{|x|}\right)^{-\gamma-1} dx\end{aligned} \tag{5}$$
   for any $u \in W_0^{2,\frac{n}{2}}(\Omega)$.

Remark 1 Sparpness
In (4) $\gamma \geq 2$ is sharp.
In (5) $\gamma \geq \frac{n}{2}$ is also sharp, and $\left(\frac{n-2}{\sqrt{n}}\right)^n$ is best constant.

Remark 2

In the proof of the noncritical case, $g(r) = r^{-2p}\left(\log \frac{R}{r}\right)^{-2}$ should be monotone decreasing and $R \geq re^{\frac{1}{p}}$. Then $K = e^{\frac{1}{p}}$.

Remark 3

In the proof of the critical case, $g^*(r) = r^{-n}\left(\log \frac{R}{r}\right)^{-\frac{n}{2}-1}$ should also be monotone decreasing and $R \geq re^{\frac{1}{2}+\frac{1}{n}}$. Moreover we need the condition to absorb the error terms in the right hand side of (5) with $C^* > 0$, hence $K^* \geq e^{\frac{1}{2}+\frac{1}{n}}$.

Remark 4

C and C^* may depend on R and γ in a weak sense. Since $g(r)$ and $g^*(r)$ tends to zero as $\gamma \to \infty$ or $R \to \infty$, therefore we can take C and C^* to be bigger.

Corollary
Let $1 < p < \frac{n}{2}$, and let

$$F_p = \Big\{f : \Omega \to \mathbb{R}^+ | f \in L^\infty_{\mathrm{loc}}(\Omega \setminus \{0\}), \quad \limsup_{|x|\to 0} |x|^{2p} f(x)\left(\log \frac{1}{|x|}\right)^2 < \infty\Big\}$$

If $f \in F_p$, then $\exists \quad \lambda(f) > 0$ such that for $u \in W_0^{2,p}(\Omega)$

$$\int_\Omega |\Delta u|^p dx \geq \Lambda_{n,p} \int_\Omega \frac{|u(x)|^p}{|x|^{2p}} dx + \lambda(f) \int_\Omega |u(x)|^p f(x) dx$$

If $f \notin F_p$, moreover if $|x|^{2p} f(x)\left(\log \frac{1}{|x|}\right)^2 \to \infty$ as $|x| \to 0$,
then no inequality of the above type can hold.

5 Application

Consider the eigenvalue problem with a singular weight

$$\begin{aligned} \Delta\left(|\Delta u|^{p-2}\Delta u\right) - \frac{\mu}{|x|^{2p}}|u|^{p-2}u &= \lambda |u|^{p-2}uf \quad \text{in} \quad \Omega \\ u = \Delta u &= 0 \qquad \text{on} \quad \partial\Omega \end{aligned} \tag{6}$$

Here $f \in \mathcal{F}_p$

$$\mathcal{F}_p = \left\{f : \Omega \to \mathbb{R}^+ | \lim_{|x|\to 0} |x|^{2p} f(x) = 0, \quad f \in L^\infty_{\mathrm{loc}}\left(\bar{\Omega} \setminus \{0\}\right)\right\}$$

$1 < p < \frac{n}{2}, 0 \leq \mu < \Lambda_{n,p}$ and $\lambda \in \mathbb{R}$.

We look for a weak solution

$$u \in W = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

We define weak solution in the following way:

Definition $u \in W$ is said to be a weak solution of (4) if for any $\phi \in C^2(\bar{\Omega})$ with $\phi = 0$ on $\partial\Omega$

$$\int_\Omega \left(|\Delta u|^{p-2} \Delta u \Delta \phi - \frac{\mu}{|x|^{2p}} |u|^{p-2} u\phi \right) dx = \lambda \int_\Omega |u|^{p-2} u f \phi dx.$$

Lemma For $u \in W \quad \exists \quad v \in W$ such that $v > 0$ and

$$\frac{\int_\Omega |\Delta u|^p dx - \mu^* \int_\Omega \frac{|u|^p}{|x|^{2p}} dx}{\int_\Omega |u|^p f dx} \geq \frac{\int_\Omega |\Delta v|^p dx - \mu^* \int_\Omega \frac{|v|^p}{|x|^{2p}} dx}{\int_\Omega |v|^p f dx}.$$

Theorem

For all $1 < p < \frac{n}{2}, \quad 0 \leq \mu < \Lambda_{n,p}$, the above problem admits a positive weak solution $u \in W$ corresponding to the first eigenvalue $\lambda = \lambda_\mu^1(f) > 0$. Moreover, as $\mu \to \Lambda_{n,p}$,

$$\text{If} \quad \lim_{|x| \to 0} |x|^{2p} f(x) = 0, \quad \Rightarrow \quad \lambda_\mu^1(f) \to \lambda(f) \geq 0$$

$$\text{If} \quad \limsup_{|x| \to 0} |x|^{2p} f(x) \left(\log \frac{1}{|x|} \right)^2 < \infty, \quad \Rightarrow \quad \lambda(f) > 0$$

$$\text{If} \quad \lim_{|x| \to 0} |x|^{2p} f(x) \left(\log \frac{1}{|x|} \right)^2 = \infty, \quad \Rightarrow \quad \lambda(f) = 0$$

6 Further results

Resent results due to H. Ando and T. Horiuchi

Higher order case 1:

$$\begin{aligned}\int_\Omega |\Delta u(x)|^2 dx \geq \left(\frac{n(n-4)}{4} \right)^2 \int_\Omega \frac{u(x)^2}{|x|^4} dx + \\ \frac{N^2 - 4N + 8}{8} \int_\Omega \frac{u(x)^2}{|x|^4} \Big[A_1(|x|)^{-2} + \\ \Big(A_1(|x|) A_2(|x|) \Big)^{-2} + \cdots + \\ + \cdots\cdots\cdots + \\ + \Big(A_1(|x|) A_2(|x|) \ldots A_k(|x|) \Big)^{-2} \Big] dx.\end{aligned}$$

Case 2

$$\begin{aligned}\int_\Omega |\Delta^2 u(x)|^2 dx \geq C_1 \int_\Omega \frac{u(x)^2}{|x|^8} dx + \\ C_2 \int_\Omega \frac{u(x)^2}{|x|^4} A_1(|x|)^{-2} \, dx + \cdots\end{aligned}$$

Resent results of missing terms for large p due to T. Horiuchi

Assume that $p > N$.

$$\int_\Omega |\nabla u(x)|^p dx \geq \left(\frac{p-N}{p}\right)^p \int_\Omega \frac{|u(x)-u(0)|^p}{|x|^p} dx + C\int_\Omega \frac{|u(x)-u(0)|^2}{|x|^p} A_1(|x|)^{-2}\, dx + \cdots$$

Conjecture ($p > \frac{N}{2}$)

$$\int_\Omega |\Delta u(x)|^p dx \geq \left(\frac{n-2p}{p}\right)^p \left(\frac{np-n}{p}\right)^p \int_\Omega \frac{|u(x)-u(0)|^p}{|x|^{2p}} dx + C\int_\Omega \frac{|u(x)-u(0)|^2}{|x|^{2p}} A_1(|x|)^{-2}\, dx + \cdots$$

7 Fundamental Facts

Fundamental facts and lemmas which are needed to prove the main results :

Rearrangement of domain and functions

Define a ball Ω^* such that $|\Omega^*| = |\Omega|$ with center at the origin. u^*: denote the rearrangement of a measurable function u

Lemma (Talenti)

Let Ω be a domain on $\mathbb{R}^n$, $n \geq 3$ and $f \in C_0^\infty(\Omega)$.

If u is the weak solution of Dirichlet problem $-\Delta u = f$

in Ω, $u|_{\partial\Omega} = 0$; v is the weak solution of Dirichlet problem

$-\Delta v = |f|^*$ in Ω^*, $v|_{\partial\Omega^*} = 0$; then $v \geq |u|^*$ pointwise.

¿From this lemma we can show

$$\int_\Omega |\Delta u|^p dx = \int_{\Omega^*} |\Delta v|^p dx$$

$$\int_\Omega \frac{|u|^p}{|x|^{2p}} dx \leq \int_{\Omega^*} \frac{|v|^p}{|x|^{2p}} dx$$

Hence we have

$$\frac{\int_\Omega |\Delta u|^p dx}{\int_\Omega \frac{|u|^p}{|x|^{2p}} dx} \geq \frac{\int_{\Omega^*} |\Delta v|^p dx}{\int_{\Omega^*} \frac{|v|^p}{|x|^{2p}} dx}$$

¿From this we can assume that u is radial and Ω is a ball.

Lemma 1. ([ACR])

For any $R > 1$, the inequality

$$\int_0^1 |h'(r)|^2 r dr \geq \frac{1}{4}\int_0^1 \frac{|h(r)|^2}{(\log\frac{R}{r})^2}\frac{dr}{r}$$

holds for any $h \in C^1(0,1)$, with $h(0) = h(1) = 0$.

Lemma 2.
For any $R > 1$, and $\beta \le 0, \alpha \in (0,1)$

$$\int_0^1 |h'(r)|^2 \left(\log \frac{R}{r}\right)^\beta r dr \ge \alpha(1-\alpha-\beta)\int_0^1 |h(r)|^2 \left(\log \frac{R}{r}\right)^{\beta-2} \frac{dr}{r}$$

holds for any $h \in C^1(0,1)$ with $h(0) = h(1) = 0$.

Lemma 3
Assume $f(r) \in C^2(B_1), f(r) > 0, \Delta f(r) \le 0$. For $u(r) = f(r)v(r) \quad r = |x|$,

$$\begin{aligned}\int_{B_1} |\Delta u|^{\frac{n}{2}} dx \ge\ & \frac{n(n-2)}{4}\omega_n \int_0^1 (v'(r))^2 v^{\frac{n-4}{2}}(r) r^{n-1} |\Delta f(r)|^{\frac{n-2}{2}} f(r) dr \\ &+ \omega_n \int_0^1 v^{\frac{n}{2}}(r)\Big\{ r^{n-1}|\Delta f(r)|^{\frac{n}{2}} + \\ &\partial_r \Big[r^{n-1}\Big(|\Delta f(r)|^{\frac{n-2}{2}} f'(r) - \partial_r |\Delta f(r)|^{\frac{n-2}{2}} f(r)\Big)\Big]\Big\} dr\end{aligned}$$

for any $u \in C_0^2(B_1)$

Lemma 4.(critical case)
For any $R > 1$ and $u \in C_0^2(B_1)$,

$$\int_{B_1} |\Delta u|^{\frac{n}{2}} dx \ge \left(\frac{n-2}{\sqrt{n}}\right) \int_{B_1} \frac{|u(x)|^{\frac{n}{2}}}{|x|^n}\left(\log \frac{R}{|x|}\right)^{-\frac{n}{2}} dx + C^* \int_{B_1} \frac{|u(x)|^{\frac{n}{2}}}{|x|^n}\left(\log \frac{R}{|x|}\right)^{-\frac{n}{2}-1} dx$$

Sketch of proofs of lemma 4:
For $u(r) = f(r)v(r)$, we set $f(r) = \left(\log \frac{R}{r}\right)^a, 0 < a < 1$. Then use lemma 3 and lemma 2.

8 A Sketch of Proof of Theorem 2

Case $1 < p < \frac{n}{2}$: (noncritical case)

We shall prove the inequality for $\gamma = 2$.

$$\int_\Omega |\Delta u|^p dx \ge \left(\frac{n-2p}{p}\right)^p \left(\frac{np-n}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x|^{2p}} dx$$

$$+ C \int_\Omega \frac{|u(x)|^p}{|x|^{2p}} \left(\log \frac{R}{|x|}\right)^{-\gamma} dx$$

For $u \in C_0^2(B_1), u > 0$, radially nonincreasing , we define

$$v(r) = u(r)r^{\frac{n}{p}-2} \quad r = |x|.$$

then

$$\Delta\left(u(r)\right) = \Delta\left(v(r)r^{2-\frac{n}{p}}\right) = r^{2-\frac{n}{p}}\left(\Delta_\beta\left(v(r)\right) - \alpha\frac{v(r)}{r^2}\right)$$

where

$$\Delta_\beta(v(r)) = v''(r) + \frac{\beta-1}{r}v'(r),$$

$$\beta = n + 4 - \frac{2n}{p} \quad \text{and} \quad \alpha = \frac{(n-2p)(np-n)}{p^2}.$$

then apply these to

$$\int_{B_1} |\Delta u|^p dx - \left(\frac{n-2p}{p}\right)^p \left(\frac{np-n}{p}\right)^p \int_{B_1} \frac{|u(x)|^p}{|x|^{2p}} dx$$

and use lemma 1 to get

$$\int_{B_1} |\Delta u|^p dx \geq \left(\frac{n-2p}{p}\right)^p \left(\frac{np-n}{p}\right)^p \int_{B_1} \frac{|u(x)|^p}{|x|^{2p}} dx$$

$$+ C\int_{B_1} \frac{|u(x)|^p}{|x|^{2p}} \left(\log\frac{R}{|x|}\right)^{-\gamma} dx$$

For $\epsilon > 0$ sufficiently small, let us define

$$u_\epsilon = \begin{cases} 0, & 0 < r < \epsilon^2 \\ \dfrac{\log\frac{r}{\epsilon^2}}{r^{\frac{n-p}{p}}\log\frac{1}{\epsilon}}, & \epsilon^2 < r < \epsilon \\ \dfrac{\log\frac{1}{r}}{r^{\frac{n-p}{p}}\log\frac{1}{\epsilon}}, & \epsilon < r < 1 \end{cases}$$

Let $w_\epsilon = \int_r^1 u_\epsilon(\rho)d\rho$. Direct calculation gives

$$w_\epsilon = \begin{cases} \left(\frac{p}{n-2p}\right)^2 \frac{1-2\epsilon^{2-\frac{n}{p}}+\epsilon^{2(2-\frac{n}{p})}}{\log\frac{1}{\epsilon}}, & 0 < r < \epsilon^2 \\ \frac{p}{n-2p}\frac{r^{2-\frac{n}{p}}\log\frac{r}{\epsilon^2}}{\log\frac{1}{\epsilon}} + \left(\frac{p}{n-2p}\right)^2 \frac{1-2\epsilon^{2-\frac{n}{p}}+r^{2-\frac{n}{p}}}{\log\frac{1}{\epsilon}}, & \epsilon^2 < r < \epsilon \\ \frac{p}{n-2p}\frac{r^{2-\frac{n}{p}}\log\frac{1}{r}}{\log\frac{1}{\epsilon}} + \left(\frac{p}{n-2p}\right)^2 \frac{1-r^{2-\frac{n}{p}}}{\log\frac{1}{\epsilon}}, & \epsilon < r < 1 \end{cases}$$

$$\Delta w_\epsilon =$$

$$\begin{cases} 0, & 0 < r < \epsilon^2 \\ \frac{1}{p}\left\{r^{-\frac{n}{p}}\left(-p + n(1-p)\log\frac{r}{\epsilon^2}\right)\right\}\left(\log\frac{1}{\epsilon}\right)^{-1}, & \epsilon^2 < r < \epsilon \\ \frac{1}{p}\left\{r^{-\frac{n}{p}}\left(p + n(1-p)\log\frac{1}{r}\right)\right\}\left(\log\frac{1}{\epsilon}\right)^{-1}, & \epsilon < r < 1 \end{cases}$$

Then

$$\int_{B_1} |\Delta w_\epsilon|^p dx = \frac{2}{p+1}\left(\frac{n(p-1)}{p}\right)^p \omega_n \log\frac{1}{\epsilon} + O\left(\left(\log\frac{1}{\epsilon}\right)^{-1}\right)$$

$$\int_{B_1} \frac{|w_\epsilon|^p}{|x|^{2p}} dx \geq \frac{2}{p+1} \left(\frac{p}{n-2p} \right)^p \omega_n \log \frac{1}{\epsilon} + O\left(\left(\log \frac{1}{\epsilon} \right)^{-1} \right)$$

and

$$\int_{B_1} |\Delta w_\epsilon|^p dx - \Lambda_{n,p} \int_B \frac{|w_\epsilon|^p}{|x|^{2p}} dx \leq O\left(\left(\log \frac{1}{\epsilon} \right)^{-1} \right)$$

also

$$\int_{B_1} \frac{|w_\epsilon|^p}{|x|^{2p}} \left(\log \frac{R}{|x|} \right)^{-\gamma} \geq O\left(\left(\log \frac{1}{\epsilon} \right)^{1-\gamma} \right)$$

Sharpness of $\Lambda_{n,p} = \left(\frac{n-2p}{p}\right)^p \left(\frac{np-n}{p}\right)^p$

$$\lim_{\epsilon \to 0} I(w_\epsilon) = \lim_{\epsilon \to 0} \frac{\int_{B_1} |\Delta w_\epsilon|^p dx}{\int_{B_1} \frac{|w_\epsilon|^p}{|x|^{2p}} dx} \leq \Lambda_{n,p}$$

But by Hardy inequality

$$\lim_{\epsilon \to 0} I(w_\epsilon) \geq \Lambda_{n,p}$$

hence

$$\lim_{\epsilon \to 0} I(w_\epsilon) = \Lambda_{n,p}$$

Optimality of γ

Assume $0 < \gamma < 2$. Optimality will follow if we can show for a unit ball B_1 that

$$\inf_{u \in W_0^{2,p}(B_1) \setminus \{0\}} I_\gamma(u) = \inf \frac{\int_{B_1} |\Delta u|^p dx - \Lambda_{n,p} \int_{B_1} \frac{|u|^p}{|x|^{2p}} dx}{\int_{B_1} \frac{|u|^p}{|x|^{2p}} \left(\log \frac{R}{|x|} \right)^{-\gamma} dx} = 0$$

Using a family $w_\epsilon \in W_0^{2,p}(B_1)$ we have $\lim_{\epsilon \to 0} I_\gamma(w_\epsilon) = 0$ Thus optimality follow. i.e. $\gamma \geq 2$.

Case $p = \frac{n}{2}$(critical case)

We shall prove for $\gamma = \frac{n}{2}$,

$$\int_\Omega |\Delta u|^{\frac{n}{2}} dx \geq \left(\frac{n-2}{\sqrt{n}} \right)^n \int_\Omega \frac{|u(x)|^{\frac{n}{2}}}{|x|^n} \left(\log \frac{R}{|x|} \right)^{-\gamma} dx$$

$$+ C^* \int_\Omega \frac{|u(x)|^{\frac{n}{2}}}{|x|^n} \left(\log \frac{R}{|x|} \right)^{-\gamma-1} dx$$

In this case the inequality follows immediately from Lemma 4 together with the symmetrization arguments.

sharpness of $\left(\frac{n-2}{\sqrt{n}}\right)$

To show sharpness, we use the test function

$$z_\epsilon = \left(\log \frac{R}{r+\epsilon} \right)^{\frac{n-2}{n}} - \left(\log \frac{R}{1+\epsilon} \right)^{\frac{n-2}{n}}$$

then we can show

$$\lim_{\epsilon\to 0}\frac{\int_{B_1}|\Delta z_\epsilon|^{\frac{n}{2}}dx}{\int_{B_1}\frac{|z_\epsilon|^{\frac{n}{2}}}{|x|^n}\left(\log\frac{R}{|x|}\right)^{-\frac{n}{2}}dx}=\left(\frac{n-2}{\sqrt{n}}\right)^n$$

Optimality of γ

We use the same test function u_ϵ with $p=\frac{n}{2}$, and

$w_\epsilon=\int_r^1 u_\epsilon(\rho)d\rho$. Then for $0<\gamma<\frac{n}{2}$

$$\lim_{\epsilon\to 0}\frac{\int_{B_1}|\Delta w_\epsilon|^{\frac{n}{2}}dx}{\int_{B_1}\frac{|w_\epsilon|^{\frac{n}{2}}}{|x|^n}\left(\log\frac{R}{|x|}\right)^{\gamma}dx}=0$$

Thus optimality follow. i.e. $\gamma\geq\frac{n}{2}$

9 A Sketch of proof of Theorem 1

Lemma Assume $u\in C_0^2(B_1)$ is radial and $u(r)>0$ where $r=|x|$. Set

$$v_1(r)=u(r)r^{\frac{n-2}{2}}A_1(r)^{-\frac{1}{2}}$$

and

$$v_k(r)=v_{k-1}(r)A_k(r)^{-\frac{1}{2}}\quad\text{for } k\geq 2.$$

If $R\geq e_k$, then

$$\begin{aligned}&\int_{B_1}|\nabla u(x)|^2dx=\\&\left(\frac{n-2}{2}\right)^2\omega_n\int_0^1 v_k(r)^2A_1(r)A_2(r)\dots A_k(r)\frac{dr}{r}\\&\quad+\frac{\omega_n}{4}\int_0^1 v_k(r)^2A_1(r)A_2(r)\dots A_k(r)\times\\&\left[A_1(r)^{-2}+\dots+\left(A_1(r)A_2(r)\dots A_k(r)\right)^{-2}\right]\frac{dr}{r}\\&\quad+\omega_n\int_0^1 v_k'(r)^2A_1(r)A_2(r)\dots A_k(r)rdr\end{aligned}\tag{7}$$

for all $k\geq 1$.

Proof Since $R\geq e_k$, A_i is define for all $1\leq i\leq k$. Then direct calculation gives

$$|u'(r)|^2=v_k(r)^2r^{-n}A_1(r)A_2(r)\dots A_k(r)\left|\frac{n-2}{2}+C\right|^2,$$

where

$$C=\frac{1}{2}A_1(r)^{-1}+\dots+\frac{1}{2}\left(A_1(r)A_2(r)\dots A_k(r)\right)^{-1}-\frac{v_k'(r)}{v_k(r)}r.$$

Then

$$\int_{B_1} |\nabla u(x)|^2 dx = \omega_n \int_0^1 |u'(r)|^2 r^{n-1} dr$$
$$= \omega_n \int_0^1 v_k(r)^2 A_1(r) A_2(r) \dots A_k(r) \left| \frac{n-2}{2} + C \right|^2 \frac{dr}{r} \tag{8}$$

Proof of the Theorem 2.1

For $k \geq 1, \alpha > 0$ we set

$$z_k(r) = \begin{cases} A_k(r)^{-\frac{\alpha}{2}}, & 0 < r \leq \frac{1}{R} \\ A_k(\frac{1}{R})^{-\frac{\alpha}{2}} \frac{\log r}{\log \frac{1}{R}}, & \frac{1}{R} < r < 1. \end{cases}$$

For $0 < r \leq \frac{1}{R}$ we let

$$w_{k,\epsilon}(r) = \frac{\alpha}{2} \Big(A_1(r) \dots A_{k-1}(r) \Big)^{-1} A_k(r)^{\frac{-\alpha}{2}-1} \frac{r}{(r+\epsilon)^2}$$

$$\tilde{w}_{k,\epsilon}(r) = \frac{\alpha}{2} \int_0^r \Big(A_1(\rho) \dots A_{k-1}(\rho) \Big)^{-1} A_k(\rho)^{\frac{-\alpha}{2}-1} \frac{\rho}{(\rho+\epsilon)^2} d\rho$$
$$= A_k(r)^{-\frac{\alpha}{2}} \frac{r^2}{(r+\epsilon)^2} - 2 \int_0^r A_k(\rho)^{-\frac{\alpha}{2}} \frac{\rho\epsilon}{(\rho+\epsilon)^3} d\rho.$$

We note that as $\epsilon \to 0$,

$w_{k,\epsilon}(r) \to z_k'(r)$

$\tilde{w}_{k,\epsilon}(r) \to z_k(r)$

for $0 < r \leq \frac{1}{R}$.

SHARPNESS OF $\frac{1}{4}$

We shall prove using inductive argument that $\frac{1}{4}$ is sharp coefficient for every missing terms.

$$\frac{\int_{B_1} |\nabla u(x)|^2 dx - \left(\frac{n-2}{2}\right)^2 \omega_n \int_0^1 v_1(r)^2 A_1(r) \frac{dr}{r}}{\omega_n \int_0^1 v_1(r)^2 A_1(r)^{-1} \frac{dr}{r}} = \frac{1}{4} +$$
$$+ \frac{\int_0^1 v_1'(r)^2 A_1(r) r dr}{\int_0^1 v_1(r)^2 A_1(r)^{-1} \frac{dr}{r}}.$$

Then using the test function defined in (9) we get

$$\int_0^{\frac{1}{R}} z_1'(r)^2 A_1(r) r dr = \frac{\alpha}{4} A_1 \Big(\frac{1}{R}\Big)^{-\alpha} \tag{9}$$

and

$$\int_0^{\frac{1}{R}} z_1(r)^2 A_1(r)^{-1} \frac{dr}{r} = \frac{1}{\alpha} A_1 \Big(\frac{1}{R}\Big)^{-\alpha}. \tag{10}$$

Hence from (9) and (10) the ratio

$$\frac{\int_0^1 z_1'(r)^2 A_1(r) r dr}{\int_0^1 z_1(r)^2 A_1(r)^{-1} \frac{dr}{r}} \to 0$$

as $\alpha \to 0$. Thus $\frac{1}{4}$ is sharp in inequality (5) for $k = 1$.

OPTIMALITY OF THE EXPONENT 2

Assume $0 \leq \gamma < 2$. Then for $k = 1$ optimality of γ will follow if we can show that

$$\inf_{v_1(r) \in C_0^1((0,1))} I_\gamma(v_1(r)) = 0.$$

where

$$I_\gamma(v_1(r)) = \frac{\int_{B_1} |\nabla u(x)|^2 dx - \left(\frac{n-2}{2}\right)^2 \omega_n \int_0^1 v_1(r)^2 A_1(r) \frac{dr}{r}}{\omega_n \int_0^1 v_1(r)^2 A_1(r)^{1-\gamma} \frac{dr}{r}}.$$

But from (7) for $k = 1$ we have

$$I_\gamma(v_1(r)) = \frac{\frac{1}{4} \int_0^1 v_1(r)^2 A_1(r)^{-1} \frac{dr}{r} + \int_0^1 v_1'(r)^2 A_1(r) r dr}{\int_0^1 v_1(r)^2 A_1(r)^{1-\gamma} \frac{dr}{r}}.$$

Then using the test function we get

$$\int_0^{\frac{1}{k}} z_1(r)^2 A_1(r)^{1-\gamma} \frac{dr}{r} = \frac{1}{\alpha + \gamma - 2} A_1\left(\frac{1}{R}\right)^{-\alpha-\gamma+2}.$$

$$\int_0^{\frac{1}{k}} z_1'(r)^2 A_1(r) r dr \quad \text{and} \quad \int_0^{\frac{1}{k}} z_1(r)^2 A_1(r)^{-1} \frac{dr}{r}$$

are finite respectively for $\alpha > 0$. Since $0 \leq \gamma < 2$, then from (9) we have for a given test function $z_1(r)$ defined in (9), $I_\gamma(z_1(r)) \to 0$ as $\alpha \to 2 - \gamma > 0$. Therefore $\gamma \geq 2$ and 2 is optimal.

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